

TWO PROBLEMS ON CARTAN DOMAINS

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ABSTRACT. Firstly, we consider the unitary geometry of two exceptional Cartan domains $\mathfrak{R}_V(16)$ and $\mathfrak{R}_{VI}(27)$. We obtain the explicit formulas of Bergman kernel function, Cauchy-Szegő kernel, Poisson kernel and Bergman metric for $\mathfrak{R}_V(16)$ and $\mathfrak{R}_{VI}(27)$. Secondly, we give a class of invariant differential operators for Cartan domain \mathfrak{R} of dimension n : If the Bergman metric of \mathfrak{R} is

$$ds^2 = \sum_{i,j=1}^n g_{ij} dz_i d\bar{z}_j, T(z, \bar{z}) = (g_{ij})$$

and

$$L(u) = T^{-1}(z, \bar{z}) \left[\frac{\partial^2 u}{\partial z_i \partial \bar{z}_j} \right],$$

then

$$L_j(u) = \{\text{The sum of all principal minors of degree } j \text{ for } L(u)\}$$

is invariant under the biholomorphic mapping of \mathfrak{R} . Let D be the irreducible bounded homogeneous domain in C^n , $P = P(z, *)$ the Poisson kernel of D , then for any fixed $J(1 \leq j \leq n)$ one has $L_j(P^{1/j}) = 0$ iff D is a symmetric domain.

1. UNITARY GEOMETRY ON EXCEPTIONAL CARTAN DOMAINS

In 1935, E. Cartan classified all symmetric bounded domains. He proved that there exist only six types of irreducible bounded symmetric domains in \mathbb{C}^n . They can be realized as follows:

$$\begin{aligned} \mathfrak{R}_I(m, n) &= \{Z \in C^{mn} | I - Z\bar{Z}' > 0, Z - (m, n) \text{ matrix}\} \\ \mathfrak{R}_{II}p &= \{Z \in C^{p(p+1)/2} | I - Z\bar{Z}' > 0, Z - \text{symmetric matrix of degree } p\} \\ \mathfrak{R}_{III}q &= \{Z \in C^{q(q-1)/2} | I - Z\bar{Z}' > 0, Z - \text{skew symmetric matrix of degree } q\} \\ \mathfrak{R}_{IV}(n) &= \{z = (z_1, \dots, z_n) \in C^n | 1 + |zz'|^2 - 2z\bar{z}' > 0, 1 - |zz'|^2 > 0\} \end{aligned}$$

or

$$= \left\{ Z = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_{n-1} \\ z_2 & z_n & 0 & \cdots & 0 \\ z_3 & 0 & z_n & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_{n-1} & 0 & 0 & \cdots & z_n \end{pmatrix} \in C^n \mid \frac{1}{2\sqrt{-1}}(Z - \bar{Z}') > 0 \right\}.$$

Besides the above four Cartan domains, there exist two exceptional Cartan domains of dimensions 16 and 27. If we denote them by $\mathfrak{R}_V(16)$ and $\mathfrak{R}_{VI}(27)$ respectively, then

$$\mathfrak{R}_V(16) = \{(Z, U) \in C^{16} \mid \frac{1}{2\sqrt{-1}}(Z - \bar{Z}') - \frac{1}{2}(U\bar{U}' + \bar{U}U') > 0\},$$

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where

$$Z = \begin{pmatrix} z_1 & z_2 & z_3 & \cdots & z_7 \\ z_2 & z_8 & 0 & \cdots & 0 \\ z_3 & 0 & z_8 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ z_7 & 0 & 0 & \cdots & z_8 \end{pmatrix}, U = \begin{pmatrix} t \\ uQ_1 \\ \vdots \\ uQ_6 \end{pmatrix}, \quad \begin{matrix} t = (t_1, \dots, t_4) \in C^4 \\ u = (u_1, \dots, u_4) \in C^4 \end{matrix}$$

and

$$\begin{aligned} Q_1 &= I^{(4)}, \quad Q_2 = \sqrt{-1} \begin{pmatrix} I^{(2)} & 0 \\ 0 & -I^{(2)} \end{pmatrix}, \quad Q_3 = \begin{pmatrix} 0 & I^{(2)} \\ -I^{(2)} & 0 \end{pmatrix} \\ Q_4 &= \sqrt{-1} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix}, \quad Q_5 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}, \\ Q_6 &= \sqrt{-1} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \\ Q_i \overline{Q}_j' + Q_j \overline{Q}_i' &= 2\delta_{ij} I^{(4)}, \quad i, j = 1, 2, \dots, 6. \end{aligned}$$

$$\Re_{VI}(27) = [(z_{11}, z_{12}, z_{13}, z_{22}, z_{23}, z_{33}) \in C^1 \times C^8 \times C^8 \times C^1 \times C^8 \times C^1 | \frac{1}{2\sqrt{-1}}(Z - \overline{Z}') > 0],$$

where

$$Z = \begin{pmatrix} z_{11} & z_{12} & z_{13} \\ z_{12}' & z_{22}I^{(8)} & z_{23} \\ z_{13}' & z_{23}' & z_{33}I^{(8)} \end{pmatrix}, \quad z_{23} = \begin{pmatrix} zT_1 \\ \cdots \\ zT_2 \end{pmatrix}, \quad z = (z_1, \dots, z_8) \in C^8$$

and

$$\begin{aligned} T_1 &= \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & & & \\ & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & & \\ & & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \\ & & & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix}, \\ T_2 &= \begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & & \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & \\ & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \\ & & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \end{pmatrix}, \end{aligned}$$

$$T_3 = \begin{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \\ & & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \\ & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \end{pmatrix},$$

$$T_4 = \begin{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \\ & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & \end{pmatrix},$$

$$T_5 = \begin{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ & & \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \\ & \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} & \end{pmatrix},$$

$$T_6 = \begin{pmatrix} & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ & & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & \end{pmatrix},$$

$$T_7 = \begin{pmatrix} & & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ & \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} & \\ & \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} & \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & \end{pmatrix},$$

$$T_8 = \begin{pmatrix} & & & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ & & \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} & \\ & \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & & \\ \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & & & \end{pmatrix}.$$

$$T_i T'_j + T_j T'_i = 2\delta_{ij} I^{(8)}, i, j = 1, 2, \dots, 8.$$

For the first four types of Cartan domains, Hua and Lu obtained many results[2,3]. Now we consider the unitary geometry on the exceptional Cartan domains $\mathfrak{R}_V(16)$ and $\mathfrak{R}_{VI}(27)$.

I Bergman kernel function

The mapping

$$\begin{aligned} W &= A[Z - \frac{1}{2}(Z_0 + \overline{Z}'_0) - \sqrt{-1}(U\overline{U}'_0 + \overline{U}_0 U') + \frac{\sqrt{-1}}{2}(U_0 \overline{U}'_0 + \overline{U}_0 U'_0)]A' \\ R &= A(U - \overline{U}_0) \end{aligned}$$

is a holomorphic automorphism of $\mathfrak{R}_V(16)$, where

$$R = \begin{pmatrix} r \\ sQ_1 \\ \vdots \\ sQ_6 \end{pmatrix}, \quad \begin{aligned} r &= (r_1, \dots, r_4) \in C^4 \\ s &= (s_1, \dots, s_4) \in C^4 \end{aligned}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{17} \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{22} \end{pmatrix} \in R^8$$

which maps point $(Z_0, U_0) \in \mathfrak{R}_V(16)$ into point $(\sqrt{-1}I, 0)$, and

$$(A'A)^{-1} = \frac{Z_0 - \overline{Z}'_0}{2\sqrt{-1}} - \frac{1}{2}(U_0 \overline{U}'_0 + \overline{U}_0 U'_0), \quad a_{22}^{-2} = \frac{z_8^0 - \overline{z}_8^0}{2\sqrt{-1}} - u_0 \overline{u}'_0.$$

By direct calculations, we have

$$\frac{\partial(W, R)}{\partial(Z, U)} = \begin{pmatrix} a_{11}^2 & & & & \\ & a_{11}a_{22}I^{(6)} & & & \\ & & a_{22}^2 & & \\ & & & a_{11}I^{(4)} & \\ & & & & a_{22}I^{(4)} \end{pmatrix}$$

and

$$\det \left[\frac{\partial(W, R)}{\partial(Z, U)} \overline{\frac{\partial(W, R)}{\partial(Z, U)}} \right] = (a_{11}a_{12})^{24} = (a_{11}^2 a_{22}^2)^{12} / a_{22}^{120} = \det(A'A)^{12} / a_{22}^{120}.$$

Hence, the Bergman kernel function of $\mathfrak{R}_V(16)$ is given by (up to a constant factor)

$$K_V(Z, U, Z, U) = \frac{\left[\frac{1}{2\sqrt{-1}}(z_8 - \overline{z}_8) - u\overline{u}' \right]^{60}}{\det \left[(2\sqrt{-1})^{-1}(Z - \overline{Z}') - \frac{1}{2}(U\overline{U}' + \overline{U}U') \right]^{12}}$$

$$= \{(2\sqrt{-1})^{-1}(z_1 - \bar{z}_1)((2\sqrt{-1})^{-1}(z_8 - \bar{z}_8)) - \sum_{j=1}^6 [(2\sqrt{-1})^{-1}(z_{j+1} - \bar{z}_{j+1}) - \frac{1}{2}(uQ_j\bar{t}' + t\bar{Q}_j'\bar{u}')^2]\}^{-12}.$$

Mapping

$$W = A \left[Z - \frac{1}{2}(Z_0 + \bar{Z}_0') \right] A'$$

is a holomorphic automorphism of $\mathfrak{R}_{VI}(27)$, where

$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22}I^{(8)} & a_{23} \\ 0 & 0 & a_{33}I^{(8)} \end{pmatrix}, a_{23} = \begin{pmatrix} aT_1 \\ aT_2 \\ \vdots \\ aT_8 \end{pmatrix}, a = (a_1, \dots, a_8) \in R^8.$$

which maps point $Z_0 \in R_{VI}(27)$ into point $\sqrt{-1}\mathbf{I}$ and

$$(A'A)^{-1} = (2\sqrt{-1})^{-1}(Z_0 - \bar{Z}_0'), (a_{22}a_{33})^{-2} = \left(\frac{z_{22}^0 - \bar{z}_{22}^0}{2\sqrt{-1}}\right)\left(\frac{z_{33}^0 - \bar{z}_{33}^0}{2\sqrt{-1}}\right) - \left(\frac{z_0 - \bar{z}_0}{2\sqrt{-1}}\right)\left(\frac{\bar{z}_0 - z_0}{2\sqrt{-1}}\right)'.$$

By direct calculations, we have

$$\begin{pmatrix} a_{11}^2 & & & & & \\ & a_{11}a_{22}I^{(8)} & & & & \\ & & a_{11}a_{33}I^{(8)} & & & \\ & & & a_{22}^2 & & \\ & * & & & a_{22}a_{33}I^{(8)} & \\ & & & & & a_{33}^2 \end{pmatrix} = \frac{\partial W}{\partial Z}.$$

and

$$\det\left(\frac{\partial W \bar{\partial} \bar{W}}{\partial Z \bar{\partial} Z}\right) = (a_{11}a_{22}a_{33})^{36} = (a_{11}^2 a_{22}^{16} a_{33}^{16})^{18} / (a_{22}^2 a_{33}^2)^{126} \\ = (\det A'A)^{18} / (a_{22}^2 a_{33}^2)^{126}$$

So, we obtain the Bergman kernel function of $R_{VI}(27)$ as follows (up to a constant factor):

$$K_{VI}(Z, \bar{Z}) = \frac{[\frac{1}{2\sqrt{-1}}(z_{22} - \bar{z}_{22})\frac{1}{2\sqrt{-1}}(z_{33} - \bar{z}_{33}) - \frac{1}{2\sqrt{-1}}(z - \bar{z})\frac{1}{2\sqrt{-1}}(z - \bar{z})']^{126}}{\det[\frac{1}{2\sqrt{-1}}(z - \bar{z})]^{18}}.$$

2. Cauchy-Szegő Kernels and Poisson Kernels

If the point (X, V) belongs to Solov boundary of $R_V(16)$, then the (X, V) satisfies the following equation:

$$\frac{X - \bar{X}'}{2\sqrt{-1}} = \frac{1}{2}(V\bar{V}' + \bar{V}V'), V = \begin{pmatrix} u \\ vQ_1 \\ \vdots \\ vQ_6 \end{pmatrix}, u = (u_1, \dots, u_4) \in C^4, v = (v_1, \dots, v_4) \in C^4.$$

By direct calculation, the Cauchy-Szegő kernel of $R_V(16)$ is

$$H_V(Z, U; X, V) = \frac{[(z_8 - \bar{x}_8)/(2\sqrt{-1}) - u\bar{v}']^{30}}{\det[(Z - \bar{X}')/(2\sqrt{-1}) - \frac{1}{2}(U\bar{V}' + \bar{U}V')]^6}$$

(up to a constant factor).

And the Poisson kernel of $R_V(16)$ is given by (up to a constant factor)

$$P_V(Z, U; X, V) = \frac{\det[(Z - \overline{Z}')(2\sqrt{-1}) - (U\overline{U}' + \overline{U}U')/2]^6 |(z_8 - \overline{x}_8)/(2\sqrt{-1}) - u\overline{v}'|^{60}}{[(z_8 - \overline{z}_8)/(2\sqrt{-1}) - u\overline{u}']^{30} |\det[(Z - \overline{X}')(2\sqrt{-1}) - (U\overline{V}' + \overline{U}V')/2]|^{12}}$$

If X belong to the silov kernel boundary of $R_{VI}(27)$, then we have

$$X = \begin{bmatrix} X'_{11} & X_{12} & X_{13} \\ X'_{12} & X_{22}I^{(8)} & X_{23} \\ X_{13} & X'_{23} & X_{33}I^{(8)} \end{bmatrix}, X_{23} = \begin{bmatrix} XT_1 \\ \vdots \\ XT_8 \end{bmatrix}, X = (X_1 \cdots, X_8) \in \mathbf{R}^8$$

By direct calculations, the Cauchy-Szegö kernel of $R_{VI}(27)$ is given by (up to a constant factor)

$$H_{VI}(Z, X) = \frac{(z_{22} - \overline{x}_{22})/(2\sqrt{-1})(z_{33} - \overline{z}_{33})/(2\sqrt{-1}) - (z - \overline{x})/(2\sqrt{-1})[\overline{(z - \overline{x})/(2\sqrt{-1})}]^{63}}{[\det(Z - \overline{X}')/(2\sqrt{-1})]^9}.$$

And the Poisson kernel of $R_{VI}(27)$ is (up to a constant factor)

$$P_{VI}(Z, X) = \frac{\det(\frac{Z - \overline{Z}'}{2\sqrt{-1}})^9 |(\frac{z_{22} - \overline{z}_{22}}{2\sqrt{-1}})(\frac{z_{33} - \overline{z}_{33}}{2\sqrt{-1}}) - (\frac{z - \overline{x}}{2\sqrt{-1}})(\frac{\overline{z - \overline{x}}}{2\sqrt{-1}})|^{126}}{|\det(\frac{Z - \overline{X}'}{2\sqrt{-1}})|^{18} [(\frac{z_{22} - \overline{z}_{22}}{2\sqrt{-1}})(\frac{z_{33} - \overline{z}_{33}}{2\sqrt{-1}}) - (\frac{z - \overline{x}}{2\sqrt{-1}})(\frac{\overline{z - \overline{x}}}{2\sqrt{-1}})]^{63}}$$

2. A CLASS OF INVARIANT DIFFERENTIAL OPERATORS ON CARTAN DOMAINS AND THEIR SOLUTIONS

1. We consider the Cartan domain of first type

$$R_I(m, n) = (Z | I - Z\overline{Z}' > 0, Z - (m, n) \text{ matrix}).$$

It is well known that the Bergman kernel function is given by (up to a constant factor)

$$K_I(Z, \overline{Z}) = [\det(I - Z\overline{Z}')]^{-(m+n)}.$$

Let

$$(g_{j\alpha}, \overline{k\beta}) = (\frac{\partial^2 \lg K_I(Z, \overline{Z})}{\partial z_{j\alpha} \partial \overline{z}_{k\beta}}).$$

Then $(g_{j\alpha}, \overline{k\beta}) = (I - Z\overline{Z}')^{-1} \cdot X(I - Z\overline{Z}')^{-1} = T_I(Z, \overline{Z}) =$ Bergman metric matrix of $R_I(m, n)$, $ds_I^2 = \sum_{j,k=1}^m \sum_{\alpha,\beta=1}^n g_{j\alpha, \overline{k\beta}} dz_{j\alpha} d\overline{z}_{k\beta} =$ Bergman metric matrix of R_I (up to a constant factor). And

$$\det(g_{j\alpha}, \overline{k\beta}) = [\det(I - Z\overline{Z}')]^{-(m+n)} = K_I(Z, \overline{Z}).$$

Let

$$L = T_1^{-1}(Z, \overline{Z}) \frac{\partial^2}{\partial z' \partial \overline{z}}, \quad L(u) = T_1^{-1}(Z, \overline{Z}) \frac{\partial^2 u}{\partial z' \partial \overline{z}},$$

where

$$\frac{\partial}{\partial z} = \left(\frac{\partial}{\partial z_{11}}, \frac{\partial}{\partial z_{12}}, \dots, \frac{\partial}{\partial z_{1n}}, \frac{\partial}{\partial z_{21}}, \dots, \frac{\partial}{\partial z_{2n}}, \dots, \frac{\partial}{\partial z_{m1}}, \dots, \frac{\partial}{\partial z_{mn}} \right),$$

$$\frac{\partial^2}{\partial z' \partial \overline{z}} = \left(\frac{\partial}{\partial z} \right)' \left(\frac{\partial}{\partial \overline{z}} \right) = \begin{pmatrix} \frac{\partial^2}{\partial z_{11} \partial \overline{z}_{11}} & \dots & \frac{\partial^2}{\partial z_{11} \partial \overline{z}_{mn}} \\ \vdots & \dots & \vdots \\ \frac{\partial^2}{\partial z_{mn} \partial \overline{z}_{11}} & \dots & \frac{\partial^2}{\partial z_{mn} \partial \overline{z}_{mn}} \end{pmatrix}.$$

If $W = f(Z)$ is the holomorphic automorphism of $R_1(m, n)$, then

$$T_1^{-1}(Z, \bar{Z}) \frac{\partial^2 u(Z)}{\partial z' \partial \bar{z}} = \left(\frac{\partial \bar{W}}{\partial Z} \right)^{'-1} T_1^{-1}(W, \bar{W}) T_1^{-1}(Z, \bar{Z}) \frac{\partial^2 u(f^{-1}(W))}{\partial W' \partial \bar{W}} \left(\frac{\partial \bar{W}}{\partial Z} \right)' \quad (4)$$

Let

$$L_j(u) = \{\text{The sum of all principal minors of degree } j \text{ for } L(u)\}$$

then $L_j(u)$ is an invariant differential operator of $R_1(m, n)$ ($j = 1, 2, \dots, mn$).

In fact, from (4) we know that

$$T_1^{-1}(Z, \bar{Z}) \frac{\partial^2 u}{\partial z' \partial \bar{z}} \text{ is similar to } T_1^{-1}(W, \bar{W}) \frac{\partial^2 u(f^{-1}(W))}{\partial W' \partial \bar{W}}$$

Suppose

$$F(\lambda) = \det \left[\lambda I - T_1^{-1}(Z, \bar{Z}) \frac{\partial^2 u}{\partial z' \partial \bar{z}} \right]$$

is the characteristic polynomial for $T_1^{-1} \frac{\partial^2 u(Z)}{\partial z' \partial \bar{z}}$, then the coefficient of λ^{mn-j} is the $L_j(u)$ ($j = 1, 2, \dots, mn$) (up to sign \pm). But the similar matrices have the same characteristic polynomial. So $L_j(u)$ is an invariant differential operator.

Specifically,

$$L_1(u) = \text{tr} \left[T_1^{-1}(Z, \bar{Z}) \frac{\partial^2 u}{\partial z' \partial \bar{z}} \right]$$

is the Laplace-Beltrami operator of (R_1, ds_1^2) .

And

$$L_{mn}(u) = \text{tr} \left[T_1^{-1}(Z, \bar{Z}) \frac{\partial^2 u}{\partial z' \partial \bar{z}} \right]$$

is the complex Monge-Ampere operator.

2. Let

$$P_j(Z, U) = \frac{\det(I - Z\bar{Z}')^{n/j}}{|\det(I - Z\bar{U})|^{2n/j}} \quad (j = 1, 2, \dots, mn),$$

$$U\bar{U}' = I, U \text{ is a } (m, n) \text{ matrix.}$$

Then

$$L_j(p_j(Z, U)) = 0, \quad j = 1, 2, \dots, mn.$$

In fact, if we expand the $P_j(Z, U)$ around the point $Z = 0$:

$$P_j(Z, U) = \left[1 - \frac{n}{j} \text{tr}(Z\bar{Z}') + \dots \right] \left[1 + \frac{n}{j} \text{tr}(Z\bar{U}') + \dots \right] \left[1 + \frac{n}{j} \text{tr}(U\bar{Z}') + \dots \right],$$

Then, we have

$$\frac{\partial^2 P_j(Z, U)}{\partial z_{j\alpha} \partial \bar{z}_{k\beta}} \Big|_{Z=0} = \begin{cases} -\frac{n}{j} + \left(\frac{n}{j} \right)^2 |u_{j\alpha}|^2, & \text{if } (j, \alpha) = (k, \beta) \\ \left(\frac{n}{j} \right)^2 \bar{u}_{j\alpha} u_{k\beta}, & \text{if } (j, \alpha) \neq (k, \beta), \end{cases}$$

where

$$U = (u_{j\alpha}) \quad \text{and} \quad U\bar{U}' = I^{(m)}.$$

So

$$L(P_j)_{Z=0} = -\left(\frac{n}{j} \right) I^{(mn)} + \left(\frac{n}{j} \right)^2 \bar{u}' u,$$

where

$$u = (u_{11} \dots u_{1n} u_{21} \dots u_{2n} \dots u_{m1} \dots u_{mn}), \quad u\bar{u}' = m.$$

Then we have

$$L_j(P_j)|_{Z=0} = \left[C_{mn}^j \left(1 - \frac{n}{j} u\bar{u}' \right) + C_{mn}^j \frac{(n)(mn-j)}{jmn} m \right] \left(-\frac{n}{j} \right)^j = 0,$$

where $C_{mn}^j = \frac{(mn)!}{j!(mn-j)!}$.

Mapping

$$W = (AZ + B)(CZ + D)^{-1} = (\bar{A}' + Z\bar{B}')^{-1}(\bar{C}' + Z\bar{D}')$$

is a holomorphic automorphism of $R_1(m, n)$, which maps U into V and maps point $Z_0 = -A^{-1}B$ into point $W = 0$ and we have

$$I - Z_0\bar{Z}_0' = (\bar{A}'A)^{-1}, \quad I - \bar{Z}_0'Z_0 = (\bar{D}'D)^{-1}, \quad \bar{C}'\bar{D}'^{-1} = A^{-1}B.$$

Then, by direct calculations, we have

$$P_j(W, V) = \frac{P_j(Z, U)}{P_j(Z_0, U)}. \quad (j = 1, 2, \dots, mn)$$

So we have

$$L_j(P_j(Z, U))|_{Z=Z_0} = P_j(Z_0, U)L_j(P_j(W, V))|_{W=0} = 0, \quad (j = 1, 2, \dots, mn)$$

This is

$$L_j(P_j(Z, U)) = 0, \quad j = 1, 2, \dots, mn.$$

If $f(U)$ is continuous on $SR_1(m, n)$ (the Silov boundary of $R_1(m, n)$),

$$SR_1(m, n) = \{U^{m,n} | U\bar{U}' = I^{(m)}\},$$

and let

$$Y_j(Z) = \int_{SR_1(m,n)} f(U) P_j(Z, U) \dot{U}, \quad j = 1, 2, \dots, mn,$$

then we have

$$L_1(Y_1(Z)) = 0,$$

For the other Cartan domain, such as $R_{II}, R_{III}, R_{IV}, R_V$ and R_{VI} , we have similar results.

3. According to the above idea, we can obtain some other invariant differential operators and some holomorphic invariants for bounded domains in C^n . If D_1 be the bounded domain in C^n , and $w = f(z)$ be the biholomorphic mapping of D_1 , which maps D_1 onto domain D_2 in C^n . let T_i be the Bergman metric of D_i ($i = 1, 2$), and

$$R_1 = -\left(\frac{\partial^2 \log \det T_1}{\partial z_i \partial \bar{z}_j} \right),$$

$$R_2 = -\left(\frac{\partial^2 \log \det T_2}{\partial w_i \partial \bar{w}_j} \right).$$

Then we have

$$\begin{aligned} 1^\circ. \quad T_1^{-1}R_1 &= \left(\frac{\partial W}{\partial Z} \right)^{\prime -1} T_2^{-1}R_2 \left(\frac{\partial W}{\partial Z} \right)', \\ R_1 T_1^{-1} &= \left(\frac{\partial W}{\partial Z} \right) R_2 T_2^{-1} \left(\frac{\partial W}{\partial Z} \right)^{-1}, \end{aligned}$$

$$2^\circ. \quad T_1^{-1} \bar{d}(dT_1 \cdot T_1^{-1}) T_1 = \left(\frac{\partial \bar{W}}{\partial Z} \right)^{-1} T_2^{-1} \bar{d}(dT_2 \cdot T_2^{-1}) T_2 \left(\frac{\partial \bar{W}}{\partial Z} \right)'.$$

Hence

$$\Delta_j(R) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } T_1^{-1} R_1\},$$

$$\bar{\Delta}_j(R) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } R_1 T_1^{-1}\},$$

$$\Delta_j(T) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } \bar{d}(dT_1 \cdot T_1^{-1})\}$$

are invariant under the biholomorphic mapping of D_1 . ($j = 1, 2, \dots, n$) Also we have that

$$\Delta_j^N(R) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } (T_1^{-1} R_1)^N\},$$

$$\bar{\Delta}_j^N(R) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } (R_1 T_1^{-1})^N\},$$

$$\Delta_j^N(T) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } (\bar{d}(dT_1 \cdot T_1^{-1}))^N\},$$

$$\Delta_j^r(L) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } (T_1^{-1} \frac{\partial^2 u}{\partial z' \partial \bar{z}})^N\},$$

$$\bar{\Delta}_l^N(L) = \{\text{The sum of all } j \text{ by } j \text{ principal minors of } (\frac{\partial^2 u}{\partial z' \partial \bar{z}})^N T_1^{-1}\},$$

are invariant under the biholomorphic mapping of D_1 ,

where N are positive integers .

If D_1 be the homogeneous domain , then

$$-T_1^{-1} R_1 = I^{(n)},$$

so $\Delta_j(R) = \bar{\Delta}_j(R) = +C_n^j (-1)^j$. If D_1 be the irreducible Cartan domain , then

$$\bar{d}(dT_1 \cdot T_1^{-1})|_{z=0} \bar{d}dT_1 \cdot T_1^{-1}, \text{ so we have}$$

$$\Delta_1(T) = \text{Bergman metric of } D_1$$

4. Suppose D is a bounded homogeneous domain in C^n containing the origin.

Let $T(Z, \bar{Z}), K(Z, \bar{Z}), P(Z, U)$ be the Bergman metric matrix. Bergman kernel function , formal Poisson kernel of D respectively.

By a linear mapping and from the relations between $T(Z, \bar{Z}), K(Z, \bar{Z})$ and $P(Z, U)$, we can assume that

$$T|_{Z=0} = \lambda I, \left(\frac{\partial^2 \log P}{\partial z' \partial \bar{z}} \right) |_{z=0} = -\mu I, \lambda \neq 0, \mu \neq 0.$$

Let

$$L_j(\mu) = \{\text{the sum of all principal minor of degree } j \text{ for } L(\mu)\}$$

$$P_j(Z, U) = P(Z, U)^{\frac{1}{j}}$$

Then

$$L_j(P_j(Z, U)) = 0$$

if and only if

$$L_1(P(Z, U)) = 0, \quad (*)$$

where

$$j = 1, 2, \dots, n.$$

Proof We have

$$L(P_j(Z, U)) = P_j(Z, U)G(P_j(Z, U))$$

$$\text{where } G(u) = T^{-1}(Z, \bar{Z}) \left[\left(\frac{\partial^2 \log u}{\partial z' \partial \bar{z}} \right) + \left(\frac{\partial \log u}{\partial z'} \right) \left(\frac{\partial \log u}{\partial \bar{z}} \right) \right]$$

Let

$$G_j(u) = \{\text{the sum of all principal minors of degree } j \text{ for } G(u)\},$$

then $G_j(u)$ is also an invariant differential operators. ($j=1, 2, \dots, n$).

If $W = f(Z)$ be the biholomorphic automorphism of D , which maps Z_0 into O and

$f(U) = V$, then $P(Z, U) = P(Z_0, U)P(W, V)$. so it is sufficient to prove that (*) holds for $Z=0$.

Because

$$G(P_i(Z, U)) = T^{-1} \left[\frac{1}{j} \left(\frac{\partial^2 \log P}{\partial z' \partial \bar{z}} \right) + \frac{1}{j^2} \left(\frac{\partial \log P}{\partial z'} \right) \left(\frac{\partial \log P}{\partial \bar{z}} \right) \right],$$

let

$$\frac{\partial \log P}{\partial z} \Big|_{z=0} = (\alpha_1, \alpha_2, \dots, \alpha_n) = \alpha.$$

Then we have

$$\begin{aligned} G(P_j)|_{z=0} &= - \left(\frac{\lambda \mu}{j} \right) I + \frac{1}{j^2} \alpha' \bar{\alpha} \quad (\text{let } (\lambda \mu)^{-1} = b, \alpha \bar{\alpha} = a) \\ &= \left(\frac{-\lambda \mu}{j} \right) [I - \left(\frac{b}{j} \right) \alpha' \bar{\alpha}] \end{aligned}$$

There exists an unitary matrix H such that

$$I - (b/j) \alpha' \bar{\alpha} = H(I - (b/j) \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix}) H^{-1},$$

So,

$$L_j(P_j)|_{z=0} = 0 \text{ iff } X_j = 0,$$

Where

$$X_j = \{\text{the sum of all principal minors of degree } j \text{ for } [I - \begin{pmatrix} \alpha(b/j) & 0 \\ 0 & 0 \end{pmatrix}]\}.$$

But

$$X_j = C_{n-1}^{j-i} (1 - (\alpha b/j)) + C_{n-1}^j = C_n^j (n - ab)/n.$$

So

$$X_1 = 0 \text{ iff } n - ab = 0.$$

But

$$X_1 = n - ab$$

and

$$X_1 = 0 \text{ iff } C_1(P_1)|_{z=0} = 0 \text{ iff } L_1(P)|_{z=0} = 0$$

So we complete the proof.

From this we have the following theorem.

Theorem: If D be the irreducible bounded homogenous domain in C^n , then for any fixed $j(1 \leq j \leq n)$ one has

$$L_j(P_j) = 0$$

if and only if D is a symmetric domain.

Proof: If D be the irreducible symmetric domain, then from II.2, we have $L_j(P_j) = 0$.

If $L_j(P_j) = 0$, then $L_1(P) = 0$, i.e. the Poisson kernel of D is annihilated by the Laplace- Beltrami operator of D under the Bergman metric. Then from the Theorem 8 of [1], that D must be symmetric.

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